Inhomogeneous discrete-time exclusion processes

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Abstract

We study discrete time Markov processes with periodic or open boundary conditions and with inhomogeneous rates in the bulk. The Markov matrices are given by the inhomogeneous transfer matrices introduced previously to prove the integrability of quantum spin chains. We show that these processes have a simple graphical interpretation and correspond to a sequential update. We compute their stationary state using a matrix ansatz and express their normalization factors as Schur polynomials. A connection between Bethe roots and Lee-Yang zeros is also pointed out.

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Introduction

The study of systems far from equilibrium has greatly benefited from exact results obtained for interacting stochastic particle systems [6, 35, 37, 38, 43, 46] and amongst these, the asymmetric simple exclusion process (ASEP) has become a paradigm. In one-dimension, the ASEP can be solved exactly and many physical quantities (such as its phase diagram, N-point correlations, density and current fluctuations etc...) can be calculated analytically [13, 24, 25, 26, 44]. Hence, the ASEP (as well as the closely related Kardar-Parisi-Zhang equation) belongs to the class of integrable models: this exceptional property explains its solvability, its intricate algebraic structure and its relations to many different fields of mathematics (representation theory, random matrices, combinatorics...). A new field, *stochastic integrability*, at the crossroads of probability and algebra, is indeed emerging, in which the ASEP plays a distinguished role [16].

Integrability is a fragile property that is easily lost when one tries to generalize or *deform* a model: a system has to respect stringent mathematical criteria in order to be integrable (basically, the Yang-Baxter relation in the bulk and reflection identities at the boundaries [29, 45]). Our aim, in the present work, is to study integrable, one species exclusion processes with inhomogeneous and non local transition rates. We shall consider both cases of periodic (section 1) and open boundary conditions (section 2): using the transfer matrix formalism, we define discrete time stochastic processes, by stating explicitly and representing graphically the updating rules for the dynamics. Then, we determine the stationary states of these models exactly. When the inhomogeneities z_i that enter the transfer matrix are set to 1, the stationary measure coincides with the (periodic or open) TASEP one.

For the open system, we construct a matrix product Ansatz [27, 8] using the Zamolodchikov-Faddeev [51] and the Ghoshal-Zamolodchikov [33] relations, thanks to the formalism developed in [47, 22, 18]. Knowing the invariant measure, we compute analytically some physical observables (such as the local density) that will be expressed in terms of Schur polynomials. Thus, our work interrelates integrability, Matrix Ansatz and combinatorial properties of interacting particle systems and contributes to extend the class of exactly solvable models in non-equilibrium statistical mechanics. We also find an unexpected relation between the Bethe roots (solutions of the Bethe equations found in [17]) and the Lee-Yang zeros of the normalization function (see section 2.5).

1 Markov process on the periodic lattice

This section is devoted to the construction of a discrete time Markov process on the periodic lattice (the site $L+1$ is identified with the site 1). After discussing the general framework in section 1.1, we construct in section 1.2 a one-parameter family of commuting operators $t(x|\bar{z})$ (using the integrability of the model). Using these operators $t(x|\bar{z})$, we define a discrete time Markov process that can be interpreted graphically (see section 1.3). The stochastic updating rules corresponding to this process are given explicitly in section 1.4. Finally, the stationary state of the process and some of its properties are studied in section 1.5.

1.1 General framework: discrete time Markov processes

We consider a one-dimensional lattice of size L. Each site of the lattice can either be empty or carry one particle; a configuration of the system can be denoted by a L-tuple $\mathcal{C} = (\tau_1, \tau_2, \dots, \tau_L)$

where we set $\tau_j = 0$ if the site j is empty and $\tau_j = 1$ if it is occupied. The configuration space is given by $\left(\mathbb{C}^2\right)^{\otimes L} = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{L \text{ times}}$ where the j^{th} component $\mathbb{C}^2$ of the tensor space represents the j^{th} site of the lattice. A configuration of the system thus corresponds to a vector $|\tau_1 \dots \tau_L\rangle = |\tau_1\rangle \otimes \cdots \otimes |\tau_L\rangle$, where we have defined the basis vectors $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The dynamics is stochastic: the system can evolve from a configuration $\mathcal{C}'$ to another configuration $\mathcal{C}$ with the probability $M(\mathcal{C}, \mathcal{C}')$. The Markov property is satisfied: the probability $M(\mathcal{C}, \mathcal{C}')$ depends only on the two configurations $\mathcal{C}$ and $\mathcal{C}'$ and not on the previous history of the system. Let $P_t(\mathcal{C})$ be the probability for the system to be in the configuration $\mathcal{C}$ at time t. Its (discrete) time evolution is given by

$$P_{t+1}(\mathcal{C}) = \sum_{\mathcal{C}' \neq \mathcal{C}} M(\mathcal{C}, \mathcal{C}') P_t(\mathcal{C}') + \left(1 - \sum_{\mathcal{C}' \neq \mathcal{C}} M(\mathcal{C}', \mathcal{C})\right) P_t(\mathcal{C}) = \sum_{\mathcal{C}'} M(\mathcal{C}, \mathcal{C}') P_t(\mathcal{C}'), \tag{1.1}$$

where the diagonal term $M(\mathcal{C}, \mathcal{C}) = 1 - \sum_{\mathcal{C}' \neq \mathcal{C}} M(\mathcal{C}', \mathcal{C})$, is the probability of remaining in configuration $\mathcal{C}$ between t and $t + 1$. This equation can be rewritten in vector-form as follows: defining

$$|P_t\rangle = \begin{pmatrix} P_t(\ (0, \dots, 0, 0, 0)\ ) \\ P_t(\ (0, \dots, 0, 0, 1)\ ) \\ P_t(\ (0, \dots, 0, 1, 0)\ ) \\ \vdots \\ P_t(\ (1, \dots, 1, 1, 1)\ ) \end{pmatrix} = \sum_{\tau_1, \dots, \tau_L \in \{0,1\}} P_t(\ (\tau_1, \dots, \tau_L)\ )\ |\tau_1 \dots \tau_L\rangle \tag{1.2}$$

the master equation (1.1) becomes

$$|P_{t+1}\rangle = M\ |P_t\rangle. \tag{1.3}$$

A fundamental quantity is the stationary probability distribution of the process, i.e., a vector $|\mathcal{S}\rangle$ such that $M|\mathcal{S}\rangle = |\mathcal{S}\rangle$ (the existence of this vector is guaranteed because the entries of each column of M sum to 1). If the process is not reversible, that is if we do not have the detailed balance property $M(\mathcal{C}', \mathcal{C})\mathcal{S}(\mathcal{C}) \neq M(\mathcal{C}, \mathcal{C}')\mathcal{S}(\mathcal{C}')$, the task of finding an explicit expression for $|\mathcal{S}\rangle$ can be very hard. However in some cases, in particular when the Markov matrix is integrable, this stationary distribution can be calculated exactly.

1.2 Inhomogeneous transfer matrix

We now use the theory of integrable systems to construct a family of operators, $t(x|\bar{z})$, called transfer matrices, that will allow us to define stochastic updating rules. In this section, we discuss only about the integrability of the transfer matrix. Its Markov property will be discussed at the end of subsection 1.3.

The building block of the transfer matrix is the R-matrix: it acts on two sites and depends on a parameter x, called the spectral parameter. Here, we will consider the following R-matrix

$$R(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 1 & 1 - x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.4}$$

It acts in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$, where we choose the basis $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ which respectively correspond to the following lattice configurations: $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$. The R-matrix satisfies the Yang-Baxter equation:

$$R_{12}\left(\frac{x_1}{x_2}\right)\; R_{13}\left(\frac{x_1}{x_3}\right)\; R_{23}\left(\frac{x_2}{x_3}\right) \;=\; R_{23}\left(\frac{x_2}{x_3}\right)\; R_{13}\left(\frac{x_1}{x_3}\right)\; R_{12}\left(\frac{x_1}{x_2}\right)\;, \tag{1.5}$$

where the subscripts denote on which tensor space the matrix R has a non trivial action: for instance $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$ etc... The R-matrix obeys the regularity relation:

$$R(1) = P \tag{1.6}$$

where P is the permutation operator ($Pv \otimes w = w \otimes v$). It also satisfies the unitarity relation:

$$R_{12}\,(x)\, R_{21}\left(\frac{1}{x}\right) = 1, \tag{1.7}$$

where $R_{21}(x) = P_{12}R_{12}(x)P_{12}$. The R-matrix (1.4) is usually associated with the continuous time TASEP [22].

From the R-matrix, we build an inhomogeneous periodic transfer matrix as follows

$$t(x|\bar{z}) = tr_0\left[R_{0,L}\left(\frac{x}{z_L}\right) R_{0,L-1}\left(\frac{x}{z_{L-1}}\right) \dots R_{0,1}\left(\frac{x}{z_1}\right)\right]. \tag{1.8}$$

The parameter x is the spectral parameter and the parameters $z_1, \dots, z_L$ are called inhomogeneities. The homogeneous case (*i.e.* $z_i = 1$) was studied in [34].

The main feature of the transfer matrix is that, due to the Yang-Baxter equation (1.5), it commutes for different values of the spectral parameter. This is the key to integrability [29]:

$$[t(x|\bar{z}), t(y|\bar{z})] = 0\,.$$

We now introduce the following operator

$$\begin{aligned} M(x|\bar{z}) &= t(x|\bar{z})t(z_1|\bar{z})^{-1} \\ &= tr_0\left[R_{0,L}\left(\frac{x}{z_L}\right) R_{0,L-1}\left(\frac{x}{z_{L-1}}\right) \dots R_{0,1}\left(\frac{x}{z_1}\right)\right] R_{2,1}(\frac{z_2}{z_1}) \dots R_{L,1}(\frac{z_L}{z_1}). \end{aligned} \tag{1.9}$$

Note that we have normalized $t(x|\bar{z})$ using $t(z_1|\bar{z})$, but a different choice $t(z_j|\bar{z})$, $j = 2, 3, ..., L$ leads to a similar rotated matrix $M(x|\bar{z})$. Obviously $M(x|\bar{z})$ commutes with $t(y|\bar{z})$ and has the same eigenvectors. We choose below to use the "normalized matrix" $M(x|\bar{z})$ instead of $t(x|\bar{z})$ because it allows one to construct easily local Hamiltonians (see e.g. remark below).

Remark: The operator $M(x|\bar{z})$ is related to the continuous time TASEP. Taking the homogeneous limit $z_i \to 1$, we are left with $M(x) = t(x)t(1)^{-1}$, where $t(x) = tr_0(R_{0,L}(x) \dots R_{0,1}(x))$. By standard computation, we obtain

$$M'(1) = \sum_{k=1}^{L} P_{k,k+1}.R'_{k,k+1}(1), \tag{1.10}$$

with the convention $L+1 \equiv 1$. The operator

$$-P.R'(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.11}$$

is the local jump operator of the continuous time TASEP. Hence $-M'(1)$ is the Markov matrix of the continuous time TASEP. Therefore, all the results obtained below for the discrete time process will be valid, when $z_i \to 1$, for the continuous time TASEP.

1.3 Graphical representation

The R-matrix and the transfer matrices defined above have a useful graphical illustration that we now explain.

R-matrix. The action of the R-matrix $R_{ij}(x/y)$ can be represented by the vertices given in fig. 1. Each vertex corresponds to a matrix element of the R-matrix. A dashed incoming line

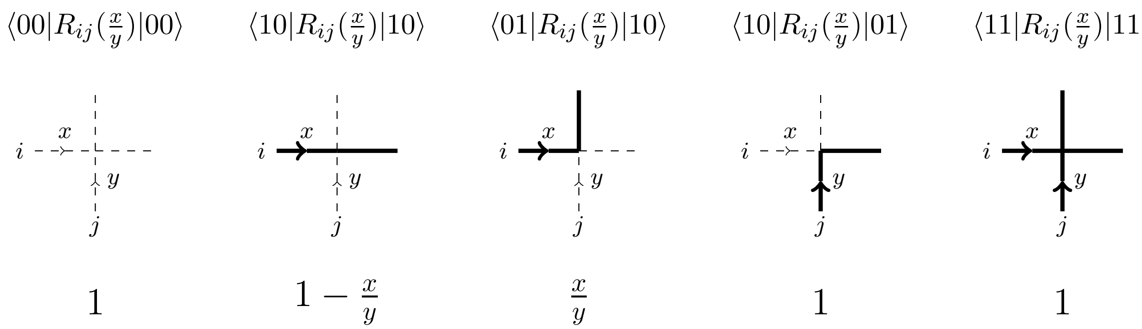


Figure 1: Non-vanishing vertices associated with R.

(according to the arrows direction) denotes the vector $|0\rangle$ (or equivalently an empty site), whereas a continuous thick line denotes the vector $|1\rangle$ (or equivalently an occupied site). In a similar way, the out-going lines (after the crossing point) represent the states $\langle 0|$ and $\langle 1|$ on which we are contracting the matrix R. The missing vertices in fig. 1 correspond to vanishing matrix elements of R. Remark that the number of particles is conserved: in every non vanishing vertex, the number of incoming continuous thick lines is equal to the number of out-going continuous thick lines.

With this graphical interpretation, we will be able to compute efficiently a matrix element of a product of R matrices acting in different components of the tensor product.

Transfer matrix. From expression (1.9) we can deduce a graphical representation for $M(x|\bar{z})$. The starting point is the lattice illustrated in fig. 2 that one has to fill according to the matrix element one wants to compute. Instead of explaining it in full generality, we take below a concrete example.

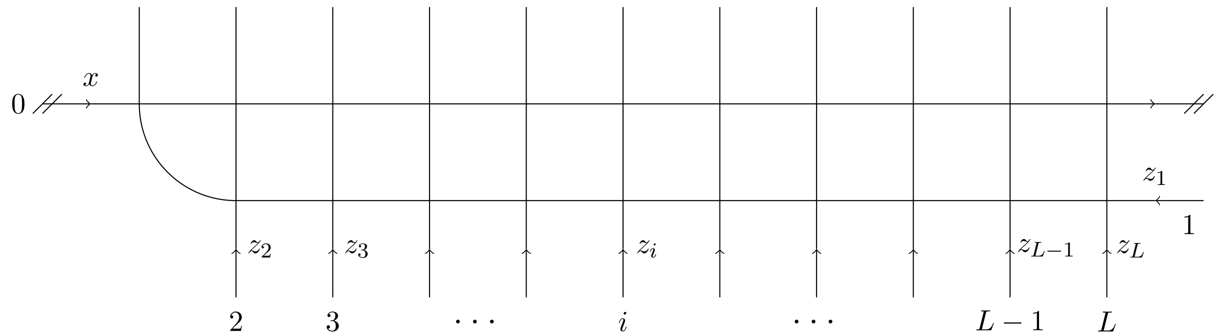


Figure 2: Graphical representation for $M(x|\bar{z})$. Since the lattice is periodic, the two double slash should be considered as linked.

As an example we take $L = 4$ and use the graphical interpretation to compute the transition rate between the initial configuration $(1, 1, 1, 0)$ and the final configuration $(1, 1, 0, 1)$. The initial (resp. final) configuration fixes the form of the incoming (resp. out-going) external lines (dashed or thick) as in fig. 3. Then, we look for drawings of the form given in fig. 3 where the remaining

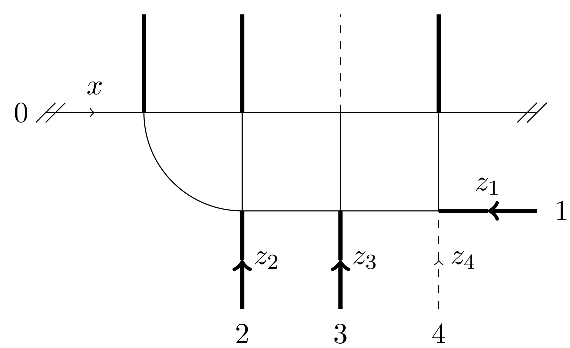


Figure 3: Starting point for the computation of $\langle 1101|M(x|\bar{z})|1110\rangle$.

thin lines have to be replaced by thick or dashed lines in such a way that the weights (as given in fig. 1) of all the vertices do not vanish. The total weight of a given possible drawing is then the product of all these weights. It is easy to see that there are only two possible drawings, given in fig. 4 together with their corresponding weights. Finally the weight of $\langle 1101|M(x|\bar{z})|1110\rangle$ is the sum of the weights of the possible drawings.

Using this graphical interpretation, we are able to compute all the possible rates between any two configurations. We remark in particular that the number of particles is conserved (as mentioned previously each non-vanishing vertex preserves the number of particles). Therefore, we restrict ourselves to a given sector with a fixed number of particles. We can also show that all the rates starting from a given configuration (with at least one particle) sum to one which proves that $M(x|\bar{z})$ can be used as a Markov matrix in the master equation (1.3). One has to impose

also

$$0 \le \frac{z_i}{z_1} \le 1 \quad \text{and} \quad 0 \le \frac{x}{z_i} \le 1\,, \quad i = 1, 2, ...L, \tag{1.12}$$

so that the probabilities are positive and less than 1. The sector with no particle is special: its dimension is 1 and the matrix $M(x|\bar{z})$ is reduced to the scalar $1 + \prod_{i=1}^{L}(1 - x/z_i)$. Therefore, it cannot be considered as a Markov matrix in the empty sector. From now, we consider only the cases with at least one particle.

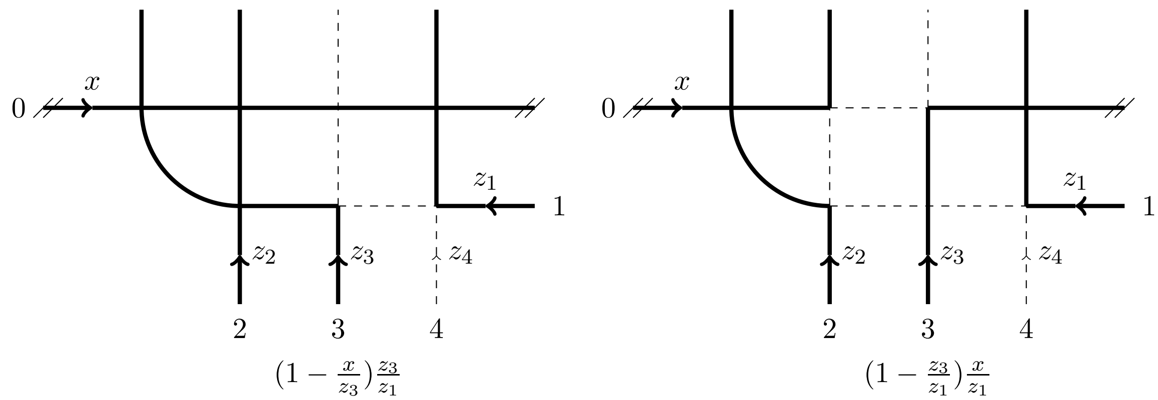


Figure 4: The two different drawings involved in the computation of the transition rate $\langle 1101|M(x|\bar{z})|1110\rangle$ with their respective weights.

1.4 Stochastic updating rules

The Markov process given by $M(x|\bar{z})$ can be interpreted as a discrete time process with sequential update. The configuration at the time $t + 1$ is obtained from the one at time t by the following dynamics:

- **Particle update:** starting from right to left (i.e. from the site L to the site $L - 1$ and so on), a particle at the site i jumps to the right on the neighboring site with a probability $1 - z_i/z_1$ provided this site is empty. The particle does not jump with the probability z_i/z_1. We remind that we are on a periodic lattice, so that the site on the right of the site L is the site 1. Note that a particle located on site 1 does not move.

- **Hole update:** once the particle update is done, one performs the hole update. Contrarily to the particle update, we do not necessarily start and finish at the sites 1 or L. Let r be the site number of a particle (we recall that we restrict ourselves to the case with at least one particle). Starting from the site r, we go from left to right up to the site $r - 1$, using periodicity and knowing that the site on the right of the site L is the site 1. A hole at the site i jumps to the left on the neighboring site with the probability $1 - x/z_{i-1}$ provided the site is occupied. The hole may stay at site i with probability x/z_{i-1}. By convention, we set $z_0 = z_L$.

As mentioned previously, we would like to emphasize that, due to the inhomogeneities in the transfer matrix, the rates depend on the site where the particle or the hole is situated. All the

probabilities are positive and less than 1 thanks to (1.12). Let us also mention that for the homogeneous case, the update simplifies. Indeed, the first step becomes trivial: the particles do not move.

These rules are illustrated in figure 5 for a chain with 4 sites in the configuration $(1,1,1,0)$ at time t. We deduce from this figure the different possible rates between the configurations which

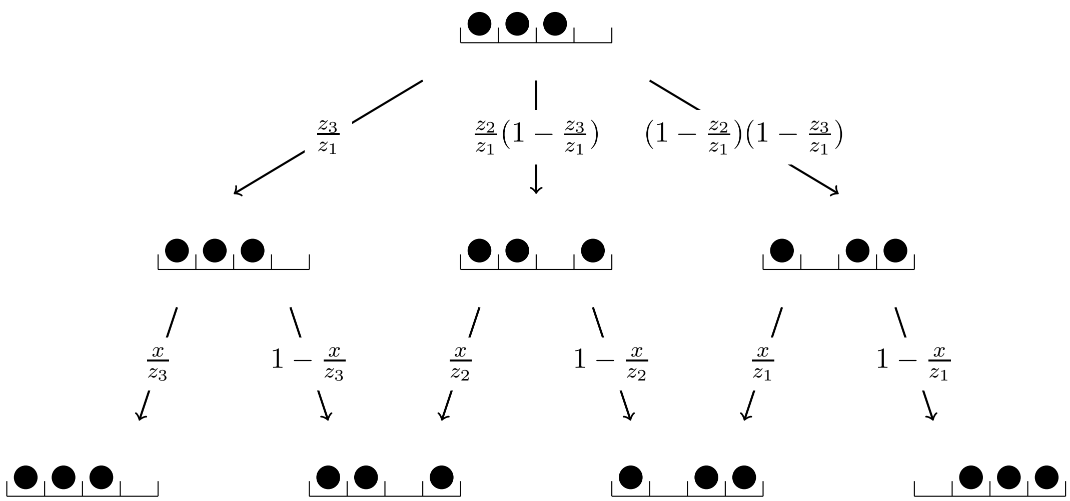


Figure 5: An example of sequential update corresponding to the Markov matrix $M(x|\bar{z})$. The first line is the configuration at time t and the third line shows the possible configurations at time $t+1$. The second line corresponds to the intermediate configurations after the update of the particles and the hole update still to be done. The label of the arrows provides the rate of the corresponding change of configurations.

correspond to the entries of $M(x|\bar{z})$. One gets

$$M(x|\bar{z})\big((1,1,0,1),(1,1,1,0)\big) = \langle 1101|M(x|\bar{z})|1110\rangle = \left(1-\frac{x}{z_3}\right)\frac{z_3}{z_1} + \left(1-\frac{z_3}{z_1}\right)\frac{x}{z_1} \qquad (1.13)$$

in accordance with the calculation done in section 1.3 (see also figure 4).

Justification of the sequential up-date. The sequential update described above can be easily identified when considering the Markov matrix $M(x|\bar{z})$ at the special point $x = z_j$. Indeed, we write $M(z_j|\bar{z}) = \Big(t(z_j|\bar{z})\mathfrak{C}\Big)\Big(\mathfrak{C}^{-1}t(z_1|\bar{z})^{-1}\Big)$ where $\mathfrak{C}$ is the cyclic permutation. Then, from the explicit expressions

$$t(z_j|\bar{z}) \;=\; R_{j,j-1}(\frac{z_j}{z_{j-1}})\cdots R_{j,1}(\frac{z_j}{z_1})\, R_{j,L}(\frac{z_j}{z_L})\cdots R_{j,j+1}(\frac{z_j}{z_{j+1}}) \qquad (1.14)$$

$$t(z_1|\bar{z})^{-1} \;=\; R_{2,1}(\frac{z_2}{z_1})\, R_{3,1}(\frac{z_3}{z_1})\cdots R_{L,1}(\frac{z_L}{z_1}) \qquad (1.15)$$

it is easy to see that $\mathfrak{C}^{-1}t(z_1|\bar{z})^{-1}$ corresponds to the particle update, while $t(z_j|\bar{z})\mathfrak{C}$ corresponds to the hole update at $x = z_j$.

Let us remark that such a simple sequential update is specific to the totally asymmetric exclusion process. For the partially asymmetric case, it would be much more involved.

1.5 Stationary State Properties

We now investigate some properties of the stationary state of the Markov chain defined by $M(x|\bar{z})$. We construct the steady state in a formal manner, then give an elementary expression for the stationary measure. Finally, we obtain some simple formulas for steady-state correlations.

1.5.1 Construction of the steady state

The building block of the stationary state is the following vector

$$v(z) = \begin{pmatrix} z \\ 1 \end{pmatrix}. \tag{1.16}$$

It satisfies the Zamolodchikov-Faddeev relation [51]

$$R_{12}(z_1/z_2)v_1(z_1)v_2(z_2) = v_1(z_1)v_2(z_2)\,. \tag{1.17}$$

The stationary state of the process $M(x|\bar{z})$ is constructed from

$$|\mathcal{S}\rangle = v_1(z_1)v_2(z_2)\dots v_L(z_L). \tag{1.18}$$

Recall that the subscripts denote which component of the tensor space the vector v belongs to. To be more precise, since the process conserves the number of particles, there are L independent sectors (as mentioned previously we do not consider the empty sector), each corresponding to a given number m of particles in the system. The stationary state is hence degenerate: there is one stationary state for each sector. Moreover, the exact normalization $Z^{(m)}$ of the stationary state depends on the sector we are considering. Then, each stationary state is given by the components of $|\mathcal{S}\rangle$ corresponding to the sector and correctly normalized.

The following calculation justifies that $|\mathcal{S}\rangle$ given in (1.18) is the stationary state of the system. Indeed, for $i = 1, 2, ..., L$ we have

$$\begin{aligned} t(z_i|\bar{z})|\mathcal{S}\rangle &= R_{i,i-1}(\frac{z_i}{z_{i-1}})\dots R_{i,1}(\frac{z_i}{z_1})R_{i,L}(\frac{z_i}{z_L})\dots R_{i,i+1}(\frac{z_i}{z_{i+1}})v_1(z_1)\dots v_L(z_L) \\ &= v_1(z_1)v_2(z_2)\dots v_L(z_L) \\ &= |\mathcal{S}\rangle. \end{aligned}$$

The first equality is obtained using the regularity property of the R-matrix (1.6) whereas the second equality is obtained using $L-1$ times the property (1.17). Clearly, $t(x|\bar{z})$ is a polynomial of degree less or equal to L in x. It is possible to show (using the graphical interpretation for instance) that in the sectors where there is at least one particle, the degree of $t(x|\bar{z})$ is in fact at most equal to $L-1$. Hence we can deduce through interpolation arguments that in these sectors, the stationary state is given by the corresponding components of $|\mathcal{S}\rangle$. Note that when we take the homogeneous limit $z_i \to 1$ we recover the uniform stationary distribution of the continuous time TASEP.

1.5.2 Calculation of steady-state averages

From the knowledge of the stationary distribution, we can calculate various physical quantities. We shall see that some observables can be expressed as symmetric polynomials in the inhomogeneity parameters $z_1, \dots, z_L$.

Probability weight of a given configuration. The weight of the configuration $(\tau_1, \dots, \tau_L)$ is readily obtained using (1.18):

$$\mathcal{S}(\tau_1, \dots, \tau_L) = \prod_{i=1}^{L} (\delta_{1,\tau_i} + z_i \delta_{0,\tau_i}) \,. \tag{1.19}$$

Normalization factor. In the sector with m particles, the normalization factor of the stationary state is obtained by summing the weights of all the configurations with m particles

$$Z^{(m)} = \sum_{\substack{I \subset \{1,\dots,L\} \\ |I| = L-m}} \prod_{i \in I} z_i \;=\; e_{L-m}(z_1, \dots, z_L) \,, \tag{1.20}$$

where e_{L-m} is the elementary symmetric homogeneous polynomial of degree $L - m$. The normalization factor can be written as a Schur polynomial:

$$Z^{(m)} = s^A_{1^{L-m}}(z_1, \dots, z_L), \quad \text{where} \quad 1^{L-m} = (\underbrace{1, ..., 1}_{L-m}, \underbrace{0, ..., 0}_{m}). \tag{1.21}$$

We remind the definition of the Schur polynomial (of type A) associated with a partition $\lambda = (\lambda_1, \dots, \lambda_L)$ with $\lambda_1 \geq \dots \geq \lambda_L \geq 0$:

$$s^A_\lambda(z_1, \dots, z_L) = \frac{\det\left((z_j)^{L-i+\lambda_i}\right)_{i,j}}{\det\left(z_j^{L-i}\right)_{i,j}}. \tag{1.22}$$

This expression of the partition function in terms of the Schur polynomial allows us to relate the value of $Z^{(m)}$ in the homogeneous limit ($z_i \to 1$) with the dimension of the representation $\pi^A(\lambda)$ of $sl(L)$ labeled by the Young tableau $[\lambda]$. Indeed, using the Weyl character formula (see e.g. [50, 32] for a review), we obtain

$$s^A_\lambda(1, \dots, 1) = \prod_{1 \leq j \leq i \leq L} \frac{\lambda_j - \lambda_i + i - j}{i - j} = \dim(\pi^A(\lambda)) \,. \tag{1.23}$$

In particular, we have

$$Z^{(m)}\Big|_{z_1 = \dots = z_L = 1} = \dim(\pi^A(1^{L-m})) = \begin{pmatrix} L \\ m \end{pmatrix} \tag{1.24}$$

in accordance with a direct computation starting from (1.20).

Density. In the sector with m particles, the particle density at site i is obtained by summing the weights of all the configurations with m particles, one of them being at site i

$$\langle n_i \rangle = \frac{e_{L-m}(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_L)}{e_{L-m}(z_1, \dots, z_L)} \,. \tag{1.25}$$

We can show that $\displaystyle\sum_{i=1}^{L} \langle n_i \rangle = m$ as expected.

Higher correlation functions. The higher correlation functions take also a very simple form. The correlations between the sites $i_1 < i_2 < \cdots < i_\ell$ is given by

$$\langle n_{i_1} n_{i_2} \dots n_{i_\ell} \rangle = \frac{e_{L-m}(z_1, \dots, z_{i_1-1}, z_{i_1+1}, \dots, z_{i_2-1}, z_{i_2+1}, \dots, z_{i_\ell-1}, z_{i_\ell+1}, \dots, z_L)}{e_{L-m}(z_1, \dots, z_L)} . \tag{1.26}$$

For $\ell > m$, the correlation functions vanish as expected since the number of particles is m and the correlation functions for more than m particles has no meaning.

2 Markov process on the open lattice with boundaries

We now consider a model on a finite lattice with two boundaries that play the role of reservoirs from which particles can enter the system or exit from it. In order to construct an integrable system, one needs to solve simultaneously the Yang-Baxter equation in the bulk, leading to the R-matrix, and Sklyanin's reflection equations at the boundaries [45], which define two boundary operators K and $\widetilde{K}$. These operators allow us to construct an inhomogeneous transfer matrix in section 2.1, that we illustrate graphically in section 2.2 and interpret as a stochastic process in section 2.3. The stationary state of this process is computed using the matrix product technique [27, 8] that allows us to calculate stationary observables (section 2.4). Using the explicit expression of the normalisation function, we relate Bethe roots and Lee-Yang zeros in section 2.5.

2.1 Inhomogeneous transfer matrix for the open system

The building blocks of the transfer matrices for the open system are the R matrix defined in section 1.4 and the following boundary matrices:

$$K(x) = \begin{pmatrix} \frac{(a+x)x}{xa+1} & 0 \\ \frac{1-x^2}{xa+1} & 1 \end{pmatrix} \quad \text{and} \quad \widetilde{K}(x) = \begin{pmatrix} \frac{1}{xb+1} & \frac{1}{xb+1} \\ 0 & \frac{xb}{xb+1} \end{pmatrix}. \tag{2.1}$$

These matrices act in the space $\mathbb{C}^2$ and we choose the basis $|0\rangle$, $|1\rangle$ corresponding to the lattice configurations (0), (1) respectively. These K-matrices are associated with the continuous time TASEP with boundaries as will be explained below.

The matrix K obeys the reflection equation [45]

$$R_{12}(\frac{x_1}{x_2})\, K_1(x_1)\, R_{21}(x_1x_2)\, K_2(x_2) = K_2(x_2)\, R_{12}(x_1x_2)\, K_1(x_1)\, R_{21}(\frac{x_1}{x_2}) . \tag{2.2}$$

The matrix $\widetilde{K}$ is related to an auxiliary $\bar{K}$ matrix, defined as $\bar{K}(x) = tr_0\left(\widetilde{K}_0(\frac{1}{x})R_{01}(\frac{1}{x^2})P_{01}\right)$, which obeys the reflection equation (2.2) associated with the matrix $\bar{R}_{12}(x) = R_{21}(1/x)$:

$$\bar{R}_{12}(\frac{x_1}{x_2})\, \bar{K}_1(x_1)\, \bar{R}_{21}(x_1x_2)\, \bar{K}_2(x_2) = \bar{K}_2(x_2)\, \bar{R}_{12}(x_1x_2)\, \bar{K}_1(x_1)\, \bar{R}_{21}(\frac{x_1}{x_2}) . \tag{2.3}$$

The explicit form of $\bar{K}$ is

$$\bar{K}(x) = \begin{pmatrix} 1 & \dfrac{(x^2-1)}{x(x+b)} \\ 0 & \dfrac{xb+1}{x(x+b)} \end{pmatrix} . \tag{2.4}$$

We can now construct the inhomogeneous open transfer matrix as follows:

$$t(x|\bar{z}) = tr_0\Big(\widetilde{K}_0(x)\, R_{0,L}(\frac{x}{z_L}) \dots R_{0,1}(\frac{x}{z_1})\, K_0(x)\, R_{1,0}(xz_1) \dots R_{L,0}(xz_L)\Big). \tag{2.5}$$

It is shown in [22] that these transfer matrices commute for different values of the spectral parameter: $[t(x|\bar{z}), t(y|\bar{z})] = 0$. Note that the derivation of this property cannot be obtained directly from the formalism developed in [45], because the matrix R does not satisfy the crossing unitarity. We use the operator $t(x|\bar{z})$ to define the following discrete time Markov process

$$|P_{t+1}\rangle = t(x|\bar{z})|P_t\rangle\ . \tag{2.6}$$

The parameters must satisfy the following constraints

$$0 \le xz_i \le 1, \quad 0 \le \frac{x}{z_i} \le 1 \quad \text{and} \qquad ax, bx \ge 0 \tag{2.7}$$

to ensure that the entries of $t(x|\bar{z})$ are probabilities. We can also show that the entries on each column of $t(x|\bar{z})$ sum to one, which guarantees the conservation of the probability $|P_t\rangle$.

Continuous time limit. In the homogeneous limit $z_i \to 1$, we are left with the operator $t(x) = t(x|\bar{z})|_{z_i=1}$. A straightforward computation gives

$$-\frac{1}{2}t'(1) = -\frac{1}{2}K_1'(1) - \sum_{k=1}^{L-1} P_{k,k+1}.R'_{k,k+1}(1) + \frac{1}{2}\bar{K}_L'(1)\ . \tag{2.8}$$

where the prime $(\,.\,)'$ stands for the derivative with respect to the spectral parameter. Then, introducing the parameters $\alpha = 1/(a+1)$ and $\beta = 1/(b+1)$, we obtain

$$-\frac{1}{2}K'(1) = \begin{pmatrix} -\alpha & 0 \\ \alpha & 0 \end{pmatrix}, \quad -P.R'(1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \frac{1}{2}\bar{K}'(1) = \begin{pmatrix} 0 & \beta \\ 0 & -\beta \end{pmatrix}\ . \tag{2.9}$$

Hence $-1/2\, t'(1)$ is nothing but the continuous time Markov matrix of the open TASEP. Therefore, the stationary state of the continuous time TASEP computed in [27] can be obtained by taking the homogeneous limit of the one computed here.

2.2 Graphical representation of the transition rates

In fig. 6, each matrix element of the K-matrix is drawn as a left-reflection diagram. The incoming lines (according to the arrows direction) before the reflection point denote the vector the matrix K is acting on. A dashed line corresponds to the vector $|0\rangle$ (or equivalently to an empty site), whereas a continuous thick line corresponds to $|1\rangle$ (or equivalently to an occupied site). In a similar way, the out-going lines (after the reflection point) represent the state of the vector with which we are contracting to the left the matrix K.

Similarly, the graphical representation for the matrix $\widetilde{K}$ is given in fig. 7.

We can now draw diagrams to represent the action of the full transfer matrix. The general picture is displayed in fig. 8.

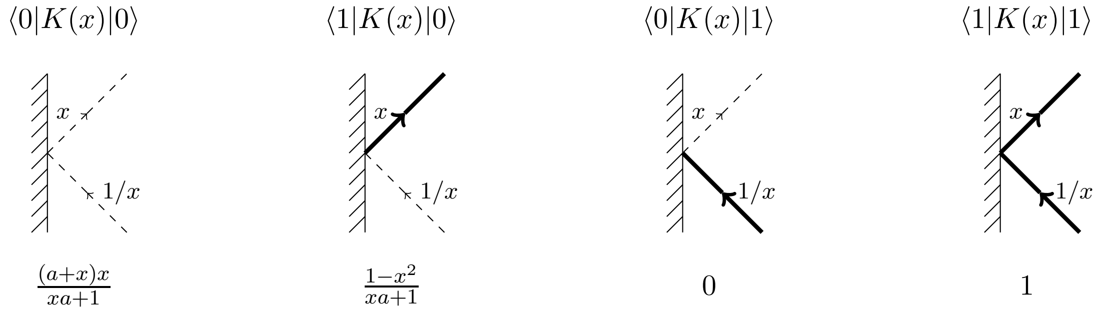


Figure 6: Graphical representation of the K-matrix.

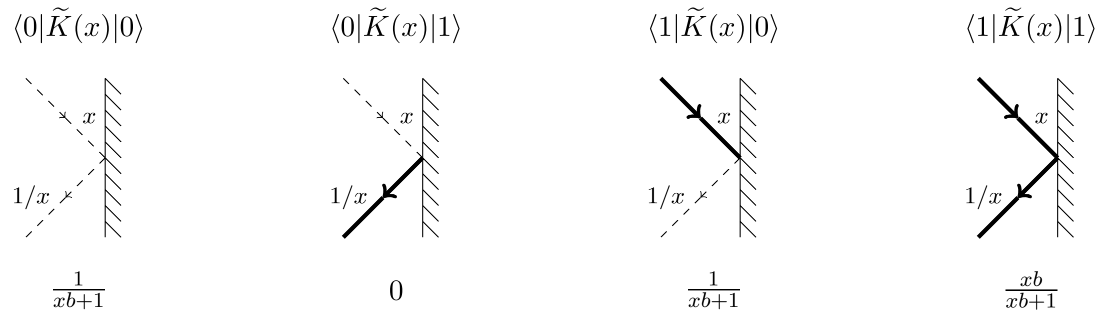


Figure 7: Graphical representation of the $\widetilde{K}$-matrix.

As an example we can compute graphically for $L = 1$ the transition rate $\langle 0|t(x|\bar{z})|1\rangle$ between the initial configuration (1) and the final configuration (0) (see fig. 9). The sum of both contributions drawn in fig. 9 gives

$$\langle 0|t(x|\bar{z})|1\rangle = \frac{x(a+x)(1-xz_1)}{(ax+1)(bx+1)} + \frac{xz_1(1-\frac{x}{z_1})}{bx+1} = (1-x^2)\frac{x(a+z_1)}{(ax+1)(bx+1)}. \tag{2.10}$$

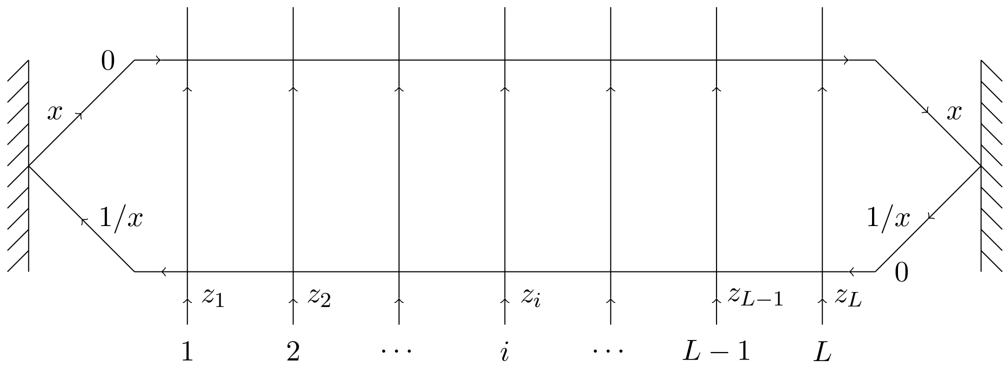
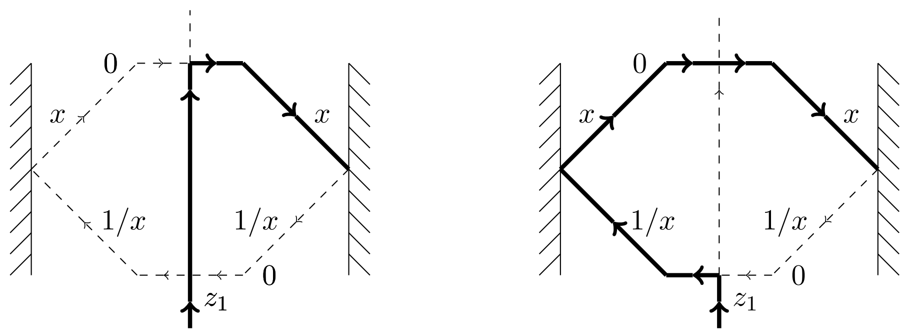


Figure 8: Graphical representation of the transfer matrix.



$\frac{x(a+x)(1-xz_1)}{(ax+1)(bx+1)}$ $\qquad$ $\frac{xz_1(1-\frac{x}{z_1})}{bx+1}$

Figure 9: Graphical computation of the transition rate $\langle 0|t(x|\bar{z})|1\rangle$ for $L = 1$. The two different contributions are represented with their respective weights.

2.3 Dynamical rules

The transfer matrix $t(x|\bar{z})$ defines a discrete time Markov process on the finite size lattice with open boundaries. The corresponding stochastic dynamics can be described explicitly using a sequential update: starting from a given configuration at time t, the configuration at time $t+1$ is obtained by the following stochastic rules

- **Initialisation:**
  - The left boundary is replaced by an additional site (the site 0 with inhomogeneity parameter 1) occupied by a particle with probability $\frac{1}{ax+1}$ and unoccupied with probability $\frac{ax}{ax+1}$.
  - The right boundary is replaced by an additional site (the site $L+1$ with inhomogeneity parameter 1) occupied by a particle with probability $\frac{bx}{bx+1}$ and unoccupied with probability $\frac{1}{bx+1}$.
- **Particle update:** starting from right to left (from site L to site 0), a particle at site i can jump to the right on the site $i+1$ (provided that the site is empty) with probability $1-xz_i$ and stay at the same place with probability xz_i.
- **Hole update:** once arrived at site 0, we go the other way starting from left to right (from site 1 to site $L+1$): a hole at site i can jump to the left on the site $i-1$ (provided that the site is occupied) with probability $1-\frac{x}{z_{i-1}}$ and stay at the same place with probability $\frac{x}{z_{i-1}}$.
- **Summation:** Then we have to drop the additional sites 0 and $L+1$ and to sum the weights corresponding to the same final configuration.

An example of such sequential update is given in fig. 10 for $L=1$. From this figure we can compute the transition rates

$$
\begin{aligned}
\langle 0|t(x|\bar{z})|1\rangle &= \frac{ax}{(ax+1)(bx+1)}\left[xz_1\left(1-\frac{x}{z_1}\right)+1-xz_1\right] \\
&\quad +\frac{1}{(ax+1)(bx+1)}\left[(1-xz_1)x^2+xz_1\left(1-\frac{x}{z_1}\right)\right] && (2.11)\\
&= (1-x^2)\frac{x(a+z_1)}{(ax+1)(bx+1)}, && (2.12)\\
\langle 1|t(x|\bar{z})|1\rangle &= \frac{x(x+b)}{bx+1}+\frac{(1-x^2)(1-xz_1)}{(ax+1)(bx+1)}. && (2.13)
\end{aligned}
$$

The equation (2.12) is, of course, identical to the expression (2.10) derived using the graphical representation in fig. 9.

2.4 Stationary state and observables

We now determine the stationary state of the inhomogeneous Markov chain, by using a matrix product representation, which will be constructed following a method developed in [22]. We shall then compute some physical characteristics of the steady-state.

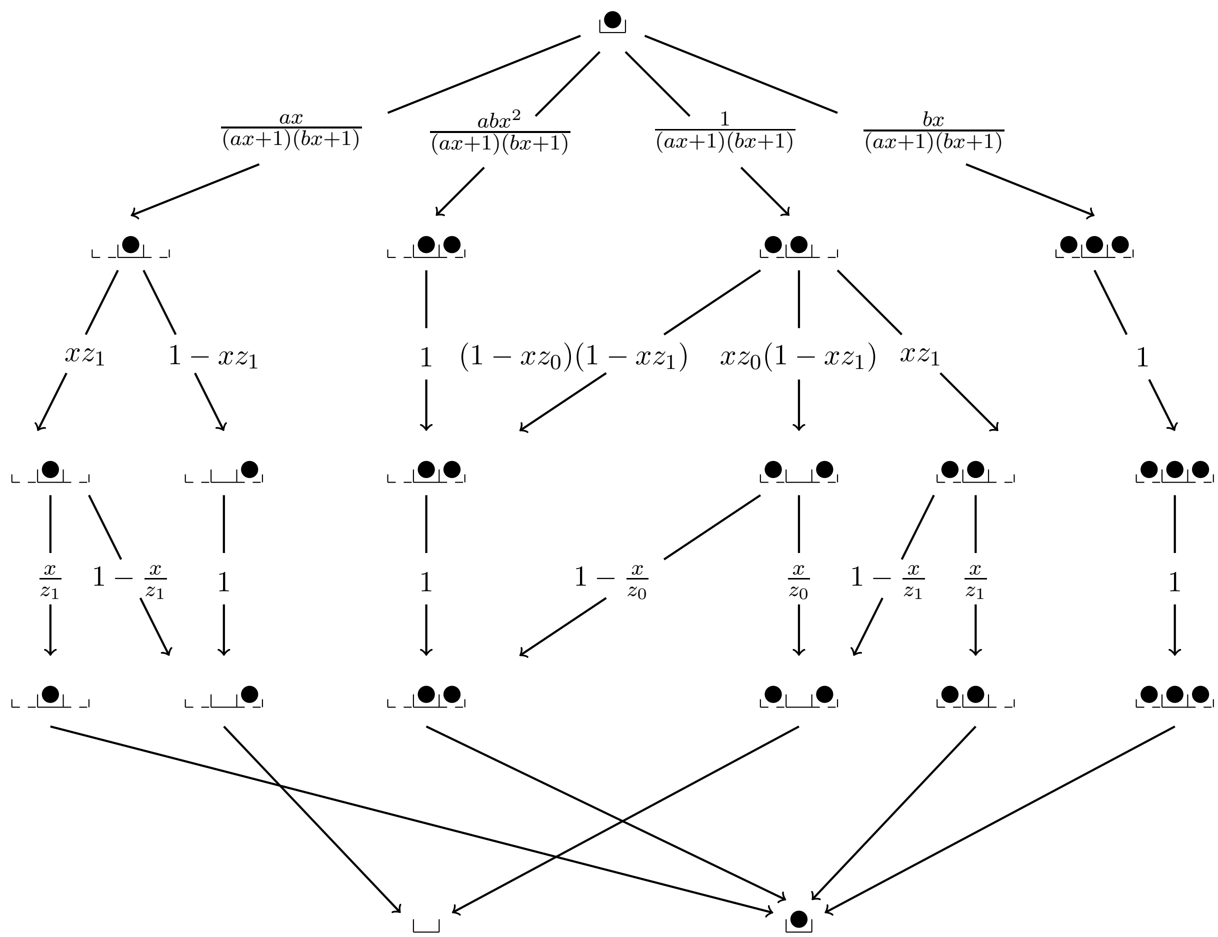


Figure 10: An example of sequential update corresponding to the Markov matrix $t(x|\bar{z})$. The first line is the configuration at time t. The second line represents the possible configurations after adding the two supplementary sites corresponding to the boundaries. The third line corresponds to the intermediate configurations after the updates of the particles. The fourth line represents the possible configurations after the updates of the holes. The label of the arrows provides the rate of the corresponding change of configurations. The last line represents the final configurations at time $t+1$ after the summation step.

2.4.1 Matrix Product Ansatz

The stationary state is expressed as a matrix product. Indeed, the probability of the configuration $\mathcal{C} = (\tau_1, \dots, \tau_L)$ can be written as

$$\mathcal{S}(\tau_1, \dots, \tau_L) = \frac{1}{Z_L} \langle\!\langle W| \prod_{i=1}^{L} ((1-\tau_i)E(z_i) + \tau_i D(z_i))|V\rangle\!\rangle \ , \tag{2.14}$$

where the normalization factor is $Z_L = \langle\!\langle W|C(z_1)\dots C(z_L)|V\rangle\!\rangle$, with $C(z) = E(z) + D(z)$. The algebraic elements $E(z)$ and $D(z)$ can be gathered in a vector

$$A(z) = \begin{pmatrix} E(z) \\ D(z) \end{pmatrix} . \tag{2.15}$$

Using this vector, the stationary state can be written in a more compact way:

$$|\mathcal{S}\rangle = \frac{1}{Z_L} \langle\!\langle W|A_1(z_1)A_2(z_2)\dots A_L(z_L)|V\rangle\!\rangle \ . \tag{2.16}$$

We now use the formalism first introduced in [47] and developed in [22, 18] to determine the algebra generated by the matrix elements of $A(z)$ and by the boundary vectors. The quadratic algebra satisfied in the bulk by E and D will be obtained using the Zamolodchikov-Faddeev relations; the boundary Ghoshal-Zamolodchikov relations show how the vectors $\langle\!\langle W|$ and $|V\rangle\!\rangle$ are coupled to E and D.

The vector $A(z)$ allows us to write easily the exchange relations between $E(z)$ and $D(z)$:

$$R_{12}(z_1/z_2)A_1(z_1)A_2(z_2) = A_2(z_2)A_1(z_1). \tag{2.17}$$

This *Zamolodchikov-Faddeev relation* [51] defines an associative algebra because the R-matrix obeys the Yang-Baxter equation. Written explicitly it gives four relations

$$\begin{cases} [E(z_1), E(z_2)] = 0 \\ [D(z_1), D(z_2)] = 0 \\ \frac{z_1}{z_2} D(z_1)E(z_2) = D(z_2)E(z_1) \\ D(z_1)E(z_2) = \frac{z_2}{z_2 - z_1} [E(z_2)D(z_1) - E(z_1)D(z_2)] \end{cases} \tag{2.18}$$

Let us remark that the third relation is implied by the fourth.

The relations between the boundary vectors $\langle\!\langle W|$, $|V\rangle\!\rangle$ and the algebraic elements $E(z)$, $D(z)$ are given by the *Ghoshal-Zamolodchikov relations*

$$\langle\!\langle W|K(z)A(1/z) = \langle\!\langle W|A(z), \qquad \bar{K}(z)A(1/z)|V\rangle\!\rangle = A(z)|V\rangle\!\rangle \ . \tag{2.19}$$

These boundary relations are consistent with the bulk exchange relations thanks to the reflection equation. Remark that the boundary vector $|V\rangle\!\rangle$ (resp. $\langle\!\langle W|$) in fact defines a class of representations (resp. dual representations) for the ZF algebra, where $|V\rangle\!\rangle$ (resp. $\langle\!\langle W|$) exists and is non-vanishing. We prove below that these classes are not empty by providing an explicit (dual) representation that belongs to them.

Written explicitly we obtain four relations

$$\left\{\begin{array}{l} \langle\!\langle W|E(z) = \langle\!\langle W|\frac{(a+z)z}{za+1}E(1/z) \\ \langle\!\langle W|E(z) = \frac{z(a+z)}{z^2-1}\langle\!\langle W|(D(1/z)-D(z)) \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} D(z)|V\rangle\!\rangle = \frac{zb+1}{z(b+z)}D(1/z)|V\rangle\!\rangle \\ D(z)|V\rangle\!\rangle = \frac{bz+1}{z^2-1}(E(z)-E(1/z))|V\rangle\!\rangle \end{array}\right. \tag{2.20}$$

Let us remark that for each boundary, the first relation is implied by the second one.

We now prove that the vector $|\mathcal{S}\rangle$ is the stationary state of the transfer matrix. Following what was done in [22], one picks up the vector $A_i(z_i)$ and moves it completely to the right using $L-i$ times the Zamolodchikov-Faddeev relation (2.17). Next, the vector $A_i(z_i)$ can be reflected against the right boundary vector $|V\rangle\!\rangle$ using the Ghoshal-Zamolodchikov relation (2.19). After that, one can push $A_i(1/z_i)$ to the left through all the $A_j(z_j)$ using $L-1$ times (2.17). Then, one can reflect $A_i(1/z_i)$ against the left boundary vector $\langle\!\langle W|$ using (2.19). Finally, one can move back $A_i(z_i)$ to its initial position to complete the cycle. Thus, we are left with:

$$t(z_i|\bar{z})|\mathcal{S}\rangle = |\mathcal{S}\rangle\ . \tag{2.21}$$

If we do the previous cycle the other way round we get

$$t(1/z_i|\bar{z})|\mathcal{S}\rangle = |\mathcal{S}\rangle\ . \tag{2.22}$$

We show below, using the graphical representation, that the numerators of the entries of $t(x|\bar{z})$ are polynomials of degree less than $L+2$. Since the equation $t(x|\bar{z})|\mathcal{S}\rangle = |\mathcal{S}\rangle$ is satisfied for $2L$ different values of x ($x = z_1, 1/z_1, \ldots, z_L, 1/z_L$), we get through interpolation arguments that for $L \geq 3$, $t(x|\bar{z})|\mathcal{S}\rangle = |\mathcal{S}\rangle$ for all x. For $L < 5$ we checked by computer that this equation is satisfied.

Degree of the numerator of $t(x|\bar{z})$: We use the graphical representation given in figures 1, 6 and 7, interpreting the last vertex of fig. 1 as in fig. 11. One can see that any matrix element

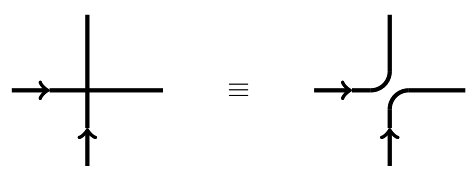


Figure 11: Vertex with non-intersecting thick lines.

$\langle\mathcal{C}'|t(x|\bar{z})|\mathcal{C}\rangle$ can be decomposed into continuous thick lines that do not intersect, as illustrated in the figure 12.

The vertices that do not enter a thick line have degree 0 in x. Thus, the total degree of the matrix element is the sum of the degree of each thick line, plus the degree coming from the boundaries that we look at separately. We first consider lines that are in the 'bulk' (that do not include a boundary). Looking at the weights of the vertices in fig. 1, one can see that the incoming part of a line always carry a degree 1. Moreover, a line corresponding to a degree n must cross at least n exit lines, the first on the left being always of degree 0. This implies that the bulk part is at most of degree L. Figure 7 shows that the right boundary does not change this counting. The left boundary can add at most a degree 2, as it can be seen on figure 6. Altogether, this leads to a total degree $L+2$.

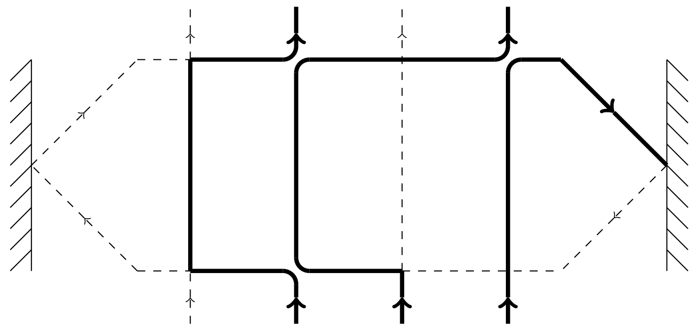


Figure 12: Decomposition of a matrix element into thick continuous lines.

Explicit representation. We now construct an explicit representation of the algebraic elements $E(z)$ and $D(z)$. We introduce the shift operators ϵ and δ such that $\delta\epsilon = 1$, which are well known in the context of the continuous time open TASEP since they enter the construction of the original matrix ansatz [27]. We define

$$\widetilde{E}(z) = z + \epsilon \quad \text{and} \quad \widetilde{D}(z) = 1/z + \delta\,, \tag{2.23}$$

and the boundary vectors such that

$$\langle\!\langle \widetilde{W}|\,\epsilon = a\,\langle\!\langle \widetilde{W}| \quad \text{and} \quad \delta|\widetilde{V}\rangle\!\rangle = b\,|\widetilde{V}\rangle\!\rangle\,. \tag{2.24}$$

It is readily verified that the Zamolodchikov-Faddeev relation (2.17) and the Ghoshal-Zamolodchikov relations (2.19) are satisfied. From the results of [27] giving explicit forms for ϵ, δ, $\langle\!\langle \widetilde{W}|$ and $|\widetilde{V}\rangle\!\rangle$, we deduce that $\langle\!\langle \widetilde{W}|$ and $|\widetilde{V}\rangle\!\rangle$ can be chosen such that $\langle\!\langle \widetilde{W}|\widetilde{V}\rangle\!\rangle \neq 0$ which guarantees the non-vanishing of $|\mathcal{S}\rangle$.

2.4.2 Steady-state probabilities

The matrix Ansatz allows us to calculate the stationary probability of any given configuration. Let $1 \leq j_1 < \cdots < j_r \leq L$ be integers and $\mathcal{C}(j_1, \ldots, j_r)$ be the configuration $(\tau_1, \ldots, \tau_L)$ with $\tau_i = 1$ if $i = j_k$ and $\tau_i = 0$ otherwise. We define the (non-normalized) weight of the word with $D(j_1), \ldots, D(j_r)$ at positions $j_1, \ldots, j_r$ as

$$\begin{aligned} W_L(j_1, \ldots, j_r) &= Z_L \times \mathcal{S}(\mathcal{C}(j_1, \ldots, j_r)) && (2.25)\\ &= \langle\!\langle W| \ldots D(z_{j_1}) \ldots D(z_{j_2}) \ldots D(z_{j_r}) \ldots |V\rangle\!\rangle, && (2.26)\end{aligned}$$

where the dots stand for $E(z_i)$ operators.

The same quantity can be calculated using the explicit representation (2.23):

$$\widetilde{W}_L(j_1, \ldots, j_r) = \langle\!\langle \widetilde{W}| \ldots \left(\delta + \frac{1}{z_{j_1}}\right) \ldots \left(\delta + \frac{1}{z_{j_2}}\right) \ldots \left(\delta + \frac{1}{z_{j_r}}\right) \ldots |\widetilde{V}\rangle\!\rangle. \tag{2.27}$$

The weight $W_L(j_1, \ldots, j_r)$ computed from the relations (2.18) and (2.20) is proportional to the weight $\widetilde{W}_L(j_1, \ldots, j_r)$ computed using the explicit representation (thanks to the unicity of the steady-state guaranteed by the Perron-Frobenius theorem). Thus, we have

$$W_L(j_1, \ldots, j_r) = f(z_1, \ldots, z_L)\widetilde{W}_L(j_1, \ldots, j_r). \tag{2.28}$$

The multiplicative coefficient is obtained by comparing the weights of the empty configuration:

$$f(z_1,\ldots,z_L)=\frac{\langle\!\langle W|E(z_1)\ldots E(z_L)|V\rangle\!\rangle}{(a+z_1)\ldots(a+z_L)\langle\!\langle W|V\rangle\!\rangle}\,; \tag{2.29}$$

$f(z_1,\ldots,z_L)$ is symmetric under the permutation of the z_i thanks to (2.18) but also under the transformation $z_i\mapsto 1/z_i$ thanks to (2.20). The group generated by these transformations is denoted by BC_L in reference to the Weyl group of the root system of the Lie algebra $sp(2L)$.

The expression of $\widetilde{W}_L(j_1,\ldots,j_r)$ is given by

$$\begin{aligned}\widetilde{W}_L(j_1,\ldots,j_r) &= \frac{z_1\ldots z_L}{z_{j_1}\ldots z_{j_r}}\sum_{p=0}^{r} b^p\sum_{q_{r-p}=j_{r-p}}^{L}\frac{1}{z_{q_{r-p}}}\sum_{q_{r-p-1}=j_{r-p-1}}^{q_{r-p}-1}\frac{1}{z_{q_{r-p-1}}}\cdots\sum_{q_1=j_1}^{q_2-1}\frac{1}{z_{q_1}}\\ &\quad\times\prod_{l_0=1}^{j_1-1}\left(1+\frac{a}{z_{l_0}}\right)\prod_{l_1=q_1+1}^{j_2-1}\left(1+\frac{a}{z_{l_1}}\right)\cdots\prod_{l_{r-p}=q_{r-p}+1}^{j_{r-p+1}-1}\left(1+\frac{a}{z_{l_{r-p}}}\right)\end{aligned} \tag{2.30}$$

By convention when $p=r$ in the first sum, there is no summation over the q_i and the formula reduces to $b^r\prod_{l_0=1}^{j_1-1}\left(1+\frac{a}{z_{l_0}}\right)$. We also set $j_{r+1}=L+1$ in the last product when $p=0$. The proof of this formula is obtained by induction on the size L, using the identity

$$\widetilde{D}(z_{j_r})\widetilde{E}(z_{j_r+1}) = \left(\frac{1}{z_{j_r}}+\delta\right)(z_{j_r+1}+\epsilon)=\frac{1}{z_{j_r}}(z_{j_r+1}+\epsilon)+z_{j_r+1}\left(\frac{1}{z_{j_r+1}}+\delta\right) \tag{2.31}$$

$$= \frac{1}{z_{j_r}}\widetilde{E}(z_{j_r+1})+z_{j_r+1}\widetilde{D}(z_{j_r+1})\,. \tag{2.32}$$

Another important step is to compute the normalization factor of the probability distribution. We define $C(z)=E(z)+D(z)$ and $\widetilde{C}(z)=\widetilde{E}(z)+\widetilde{D}(z)$. The normalization factors are thus given by $Z_L(z_1,\ldots,z_L)=\langle\!\langle W|C(z_1)\ldots C(z_L)|V\rangle\!\rangle$, and $\widetilde{Z}_L(z_1,\ldots,z_L)=\langle\!\langle \widetilde{W}|\widetilde{C}(z_1)\ldots\widetilde{C}(z_L)|\widetilde{V}\rangle\!\rangle$ for the explicit representation. Thanks to the property (2.28), we have

$$Z_L(z_1,\ldots,z_L)=f(z_1,\ldots,z_L)\widetilde{Z}_L(z_1,\ldots,z_L). \tag{2.33}$$

Z_L and $\widetilde{Z}_L$ are symmetric under BC_L.

Our goal is to get an analytic expression for the normalization factor. For this purpose, for any sequence of complex numbers $\mathbf{u}=(u_1,u_2,\ldots)$, we define the shifted product by

$$(z|\mathbf{u})^k=\begin{cases}(z-u_1)(z-u_2)\cdots(z-u_k)\,, & k>0,\\ 1\quad\text{if}\quad k=0,\end{cases} \tag{2.34}$$

We have the result (see proof in appendix A):

$$\widetilde{Z}_L(z_1,\ldots,z_L)=\frac{\det\left((z_j|\mathbf{v})^{L+2-i}-(1/z_j|\mathbf{v})^{L+2-i}\right)_{i,j}}{\det\left(z_j^{L+1-i}-(1/z_j)^{L+1-i}\right)_{i,j}}\quad\text{with}\quad \mathbf{v}=(-a,-b,0,\ldots,0). \tag{2.35}$$

When $a, b = 0$, we recognize in the L.H.S. of equation (2.35), the expression of the Schur polynomial of type C associated to the partition $1^L = (1, ..., 1)$. We remind that the Schur polynomial of type C associated with the partition $\lambda = (\lambda_1, \dots, \lambda_L)$ with $\lambda_1 \geq \dots \lambda_L \geq 0$ is defined by:

$$\widetilde{Z}_L(z_1, \dots, z_L)\Big|_{a=b=0} = s_\lambda^C(z_1, \dots, z_L) = \frac{\det\left((z_j)^{L+1-i+\lambda_i} - (1/z_j)^{L+1-i+\lambda_i}\right)_{i,j}}{\det\left(z_j^{L+1-i} - (1/z_j)^{L+1-i}\right)_{i,j}} \,. \tag{2.36}$$

As in the periodic case where the normalization factor is linked to the Schur polynomial of type A and to the representation of the Lie algebra $sl(L)$ (see (1.23) and (1.24)), the normalization factor (2.36) is given in terms of the Schur polynomial of type C and is associated to representation of the Lie algebra $sp(2L)$. These observations will allow us to use results of the Lie algebra theory to take the homogeneous limit $z_i \to 1$ (see the remark below).

When a, b are arbitrary, we need to use some generalizations of the Schur polynomial, called *shifted (or factorial) Schur polynomials*. A shifted Schur polynomial of type A, is defined, for any sequence of complex numbers $\mathbf{u} = (u_1, u_2, ...)$ [36], as follows:

$$s_\lambda^A(z_1, \dots, z_L|\mathbf{u}) \;=\; \frac{\det\left((z_j|\mathbf{u})^{L-i+\lambda_i}\right)_{i,j}}{\det\left(z_j^{L-i}\right)_{i,j}} \,. \tag{2.37}$$

Similarly, we define the shifted (or factorial) Schur polynomial of type C as[5]

$$s_\lambda^C(z_1, \dots, z_L|\mathbf{u}) = \frac{\det\left((z_j|\mathbf{u})^{L+1-i+\lambda_i} - (1/z_j|\mathbf{u})^{L+1-i+\lambda_i}\right)_{i,j}}{\det\left(z_j^{L+1-i} - (1/z_j)^{L+1-i}\right)_{i,j}} \,. \tag{2.38}$$

Thus, the normalization factor (2.35) is the shifted Schur polynomial of type C associated to the partition 1^L and to the sequence $\mathbf{v} = (-a, -b, 0, 0, \dots)$. For this particular partition, the shifted Schur polynomial of type C can be expanded on the usual type C Schur polynomials as (see proof in appendix B)

$$\widetilde{Z}_L(z_1, \dots, z_L) = s_{1^L}^C(z_1, \dots, z_L|\mathbf{v}) = \sum_{n=1}^{L+1} \frac{a^n - b^n}{a - b}\, s_{1^{L+1-n}}^C(z_1, \dots, z_L). \tag{2.39}$$

Using the explicit representation, we can also determine the particle density at site i from the stationary measure:

$$\begin{aligned} n_i(z_1, \dots, z_L) \;&=\; \left(b + \frac{1}{z_i}\right) \frac{\widetilde{Z}_{L-1}(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_L)}{\widetilde{Z}_L(z_1, \dots, z_L)} + \\ (1 - ab) \;&\times\; \sum_{k=0}^{L-i-1} \frac{\widetilde{Z}_{i-1+k}(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{i+k})\Big|_{b=0} \widetilde{Z}_{L-i-1-k}(z_{i+2+k}, \dots, z_L)\Big|_{a=0}}{\widetilde{Z}_L(z_1, \dots, z_L)}. \end{aligned}$$

[5] It is easy to see that $s_\lambda^C(z_1, \dots, z_L|\mathbf{u})$ is indeed a polynomial in the variables $z_1, \dots, z_L$.

Remark: In the homogeneous limit $z_i \to 1$, $s^C_{1^{L+1-n}}(1,\dots,1)$ is equal to the dimension of the $sp(2L)$ representation associated to the partition 1^{L+1-n}:

$$s^C_{1^{L+1-n}}(1,\dots,1) = \frac{n}{L+1}\left(\begin{array}{c}2L+2\\ L+1-n\end{array}\right) = dim\left(\pi^C(1^{L+1-n})\right). \tag{2.40}$$

This expression is obtained through the general formula (see [32] for instance):

$$dim\left(\pi^C(\lambda)\right) = \prod_{1\le i<j\le L}\left(\frac{\lambda_i-\lambda_j+j-i}{j-i}\;\frac{\lambda_i+\lambda_j+2L+2-j-i}{2L+2-j-i}\right)\prod_{i=1}^{L}\frac{\lambda_i+L+1-i}{L+1-i}.$$

Then from (2.39), we get

$$\widetilde{Z}_L(1,1,...,1) = \frac{g_L(a)-g_L(b)}{a-b} \quad \text{with} \quad g_L(x) = \sum_{n=0}^{L}\frac{n+1}{L+1}\left(\begin{array}{c}2L+2\\ L-n\end{array}\right)x^{n+1} \tag{2.41}$$

in accordance with the results known for continuous time open TASEP [27]

$$Z_L = \frac{h_L(\frac{1}{\alpha})-h_L(\frac{1}{\beta})}{\frac{1}{\alpha}-\frac{1}{\beta}} \quad \text{with} \quad h_L(x) = \sum_{p=1}^{L}\frac{p}{2L-p}\left(\begin{array}{c}2L-p\\ L\end{array}\right)x^{p+1}. \tag{2.42}$$

The equality between these two expressions is ensured by the identity

$$g_L(x-1) = h_L(x) - \frac{1}{L}\left(\begin{array}{c}2L\\ L-1\end{array}\right) \tag{2.43}$$

and recalling that $a = \frac{1}{\alpha}-1$ and $b=\frac{1}{\beta}-1$.

In particular, when $a=b=0$ (i.e. $\alpha=\beta=1$), Z_L is equal to the dimension of the representation 1^L, which is the Catalan number $C_{L+1} = \frac{1}{L+2}\left(\begin{array}{c}2L+2\\ L+1\end{array}\right)$.

2.5 Partition function and Baxter Q-operator

The models considered here have non-diagonal boundary matrices (see (2.1)) and cannot be solved by the usual integrability methods. In the last few years, various specific techniques have been developed for solving such problems, based on the functional Bethe ansatz [40, 39, 30], the coordinate Bethe ansatz [19, 20, 21, 48], the separation of variables [31, 42], the q-Onsager approach [1] and the algebraic Bethe ansatz [4, 3]. Recently, a generalization [12, 41] of the TQ relations expresses the eigenvalues of problems in terms of solutions of a new type of Bethe equations, called *inhomogeneous Bethe equations.*

In this section, we shall find an unexpected relation between the 'partition function' Z_L and the Baxter Q operator appearing in the TQ-relations.

In [17], the Bethe equations corresponding to the eigenvalues and the eigenvectors of the transfer matrix (2.5) have been computed using the modified algebraic Bethe ansatz. In this context, the eigenvalue of (2.5) corresponding to the stationary state $\Lambda(x)$ is given by

$$\Lambda(x) = x^{L+1}\frac{b+x}{bx+1}\prod_{k=1}^{L}\frac{xu_k-1}{u_k-x} - \frac{(x^2-1)}{(bx+1)}\prod_{j=1}^{L}\left[(x-z_j)(x-\frac{1}{z_j})\right]\prod_{k=1}^{L}\frac{u_k}{u_k-x}\,, \tag{2.44}$$

where u_k are called Bethe roots and are solutions of the following Bethe equations

$$\prod_{p=1}^{L} \frac{(u_j - z_p)(u_j z_p - 1)}{u_j z_p} = (u_j + b) \prod_{\substack{k=1 \\ k \neq j}}^{L} \left(u_j - \frac{1}{u_k} \right) , \qquad \text{for } j = 1, 2, \dots, L \,. \tag{2.45}$$

Let us stress that here the Bethe roots are the ones corresponding to the steady state. The Bethe roots corresponding to other states obey different Bethe equations (see [17], or [23] for the homogeneous case). We now introduce the following function:

$$Q(x) = \prod_{k=1}^{L} \left(\frac{1}{u_k} - \frac{1}{x} \right) . \tag{2.46}$$

This function (up to a coefficient x^L) is linked to the polynomial Q of Baxter [2]: its zeros are the Bethe roots. The eigenvalue (2.44) can be written in terms of the Q function as follows

$$\Lambda(x) = \frac{x(b+x)Q(1/x)}{(bx+1)Q(x)} - \frac{(x^2-1)}{(bx+1)Q(x)} \prod_{j=1}^{L} \left[(x - z_j)(\frac{1}{x z_j} - 1) \right] . \tag{2.47}$$

For the steady-state eigenvector, we know that $\Lambda(x) = 1$ (see section 2.4). Hence $Q(x)$ satisfies

$$x(x+b)Q(1/x) - (1+bx)Q(x) = (x^2-1) \prod_{j=1}^{L} \left[(x - z_j)(\frac{1}{x z_j} - 1) \right] . \tag{2.48}$$

This equation, called TQ relation, allows us to compute explicitly the function Q: for a given L, it can be shown that equation (2.48) has a unique solution of the form (2.46).

For the model studied here, the function Q and the normalization factor $\widetilde{Z}(z_1, \dots, z_L)$ are closely related. Namely, we get

$$Q(x) = \widetilde{Z}(z_1, \dots, z_L)\big|_{a \to -1/x} = \sum_{n=1}^{L+1} \frac{(-1/x)^n - b^n}{-1/x - b} \, s^C_{1^{L+1-n}}(z_1, \dots, z_L) \tag{2.49}$$

$$= \sum_{p=0}^{L} \left(-\frac{1}{x} \right)^p \sum_{n=0}^{L-p} b^n \;\; s^C_{1^{L-n-p}}(z_1, \dots, z_L) \,. \tag{2.50}$$

where we have used the explicit form of $\widetilde{Z}$ given in (2.39). To prove this result, we remark that $Q(x)$ given by (2.50) has the form (2.46) and we show that it satisfies the TQ relation (2.48). We have

$$x(x+b)Q(1/x) - (1+bx)Q(x) = -x \sum_{n=1}^{L+1} \left((-x)^n - (-1/x)^n \right) s^C_{1^{L+1-n}}(z_1, \dots, z_L) \tag{2.51}$$

$$= (x^2-1) s^C_{1^L}(z_1, \dots, z_L | x, 1/x, 0, \dots, 0) \,. \tag{2.52}$$

Finally, we readily check that $s^C_{1^L}(z_1, \dots, z_L | x, 1/x, 0, \dots, 0)$ vanishes at the points $x = z_j$ and $x = 1/z_j$ which allows us to conclude that (2.52) is equal to the L.H.S. of (2.48).

Relation (2.49) means that the Bethe roots (zeros of the function Q) are linked to the zeros of the steady-state normalization factor in the complex plane of the transition rate a. These zeros appeared previously (for the homogeneous case) in [7, 5] as Lee-Yang zeros and allows the generalization of the Lee-Yang theory for the phase transition of non equilibrium system. Therefore, relation (2.49) expresses an unexpected relation between two objects arising from very different contexts.

3 Conclusion

In this paper, we study discrete time Markov processes with periodic or open boundaries and with inhomogeneities. The transfer matrices associated with quantum spin chains are used to construct the Markov matrix. We provide a graphical representation of these processes and an interpretation in terms of sequential updates. We also study their stationary states using a matrix ansatz developed in [47, 22] and express their normalization in terms of Schur polynomials. Finally, in the case with open boundaries, we find a connection between the Bethe roots and the Lee-Yang zeros.

We believe that numerous generalizations of these works are possible and that connections with other approaches will be fruitful. For the moment, we studied the models on general grounds, without specifying the values of the inhomogeneities. However, it would be interesting to take particular choices inspired by physical problems. In these cases, we may be able to go further in the computations of the physical quantities. For example, one can choose the inhomogeneities randomly, according to a certain law, and study the behavior of the density and of the current.

To compute the stationary state, we use a generalization of the matrix ansatz but other methods have been developed to study such problems. For example, the study of qKZ equations [28, 9] seems to be closely related to the approach developed here. A common point of both methods is the use of different types of symmetric polynomials (or their deformation). Let us also mention the combinatorial interpretation of the matrix ansatz developed in [14] which may be generalized to the case with inhomogeneities.

The method presented here can be used for other models: for example, in a recent work [18], we classified integrable boundary conditions for the open two species exclusion process generalizing the model introduced in [49] (see also [10, 15]). Thus, discrete time inhomogeneous two species exclusion processes can be constructed and studied following the lines presented here. The generalization to N-species also seems possible [11].

Finally, the correspondence between the Bethe roots and the Lee-Yang zeros deserves further investigation. We do not know if it is valid only for the TASEP or if it is a general feature. Such connection may shed light on the Bethe equations and on the Lee-Yang theory for out-of-equilibrium models [7, 8].

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A Normalization factor and Schur polynomials of type C

The normalization factor can be expressed in terms of a shifted Schur polynomial of type C

$$\widetilde{Z}(z_1,\ldots,z_L) = s^C_{1^L}(z_1,\ldots,z_L|\mathbf{v}) \tag{A.1}$$

Proof Let us show be induction on L the previous formula.

- For $L=1$, we have

$$\widetilde{Z}_1(z_1) = \langle\!\langle \widetilde{W}| \left(z_1+\frac{1}{z_1}+\epsilon+\delta\right)|\widetilde{V}\rangle\!\rangle = z_1+\frac{1}{z_1}+a+b,$$

  in agreement with

$$s^C_{1^1}(z_1|\mathbf{v}) = \frac{(z_1+a)(z_1+b)-\left(\frac{1}{z_1}+a\right)\left(\frac{1}{z_1}+b\right)}{z_1-\frac{1}{z_1}} = z_1+\frac{1}{z_1}+a+b.$$

- Let $L\geq 1$ such that the property holds for $L-1$. Consider $\widetilde{Z}_L(z_1,\ldots,z_L)$ and $s^C_{1^L}(z_1,\ldots,z_L|\mathbf{v})$ as Laurent polynomials in the variable z_L. It is easy to see that

$$\begin{aligned}\widetilde{Z}_L(z_1,\ldots,z_L) \quad &= \quad \langle\!\langle \widetilde{W}| \left(z_1+\frac{1}{z_1}+\epsilon+\delta\right)\ldots\left(z_L+\frac{1}{z_L}+\epsilon+\delta\right)|\widetilde{V}\rangle\!\rangle \\ &\underset{z_L\to\infty}{\sim} \quad z_L \langle\!\langle \widetilde{W}| \left(z_1+\frac{1}{z_1}+\epsilon+\delta\right)\ldots\left(z_{L-1}+\frac{1}{z_{L-1}}+\epsilon+\delta\right)|\widetilde{V}\rangle\!\rangle \\ &\underset{z_L\to\infty}{\sim} \quad z_L\,\widetilde{Z}_{L-1}(z_1,\ldots,z_{L-1}).\end{aligned}$$

We have also $\widetilde{Z}_L(z_1,\ldots,z_L) \underset{z_L\to 0}{\sim} \widetilde{Z}_{L-1}(z_1,\ldots,z_{L-1})/z_L$. Using the determinant formula for $s^C_{1^L}(z_1,\ldots,z_L|\mathbf{v})$, we can develop the determinants in the numerator and denominator with respect to the last column to show that $s^C_{1^L}(z_1,\ldots,z_L|\mathbf{v}) \underset{z_L\to\infty}{\sim} z_L s^C_{1^{L-1}}(z_1,\ldots,z_{L-1}|\mathbf{v})$ and $s^C_{1^L}(z_1,\ldots,z_L|\mathbf{v}) \underset{z_L\to 0}{\sim} s^C_{1^{L-1}}(z_1,\ldots,z_{L-1}|\mathbf{v})/z_L$. Using the induction hypothesis we deduce that

$$s^C_{1^L}(z_1,\ldots,z_L|\mathbf{v}) \underset{z_L\to\infty}{\sim} \widetilde{Z}_L(z_1,\ldots,z_L) \quad \text{and} \quad s^C_{1^L}(z_1,\ldots,z_L|\mathbf{v}) \underset{z_L\to 0}{\sim} \widetilde{Z}_L(z_1,\ldots,z_L).$$

Hence $\widetilde{Z}_L(z_1,\ldots,z_L)$ and $s^C_{1^L}(z_1,\ldots,z_L|\mathbf{v})$ are Laurent polynomials in the variable z_L of degree 1 in z_L and $1/z_L$ and they have the same coefficients in front of z_L and $1/z_L$. So we can write

$$\widetilde{Z}_L(z_1,\ldots,z_L) - s^C_{1^L}(z_1,\ldots,z_L|\mathbf{v}) = f(z_1,\ldots,z_{L-1}),$$

with f independent of z_L. But $\widetilde{Z}_L(z_1,\ldots,z_L)$ and $s^C_{1^L}(z_1,\ldots,z_L|\mathbf{v})$ are symmetric under the permutation of the z_i's, so f cannot depend of the z_i's and hence is constant. We compute this constant by evaluating the previous expression at the particular point where all the z_i's are equal to 1. On one hand the value of $s^C_{1^L}(1,\ldots,1|\mathbf{v})$ can be computed using (2.39) and (2.40). On the other hand the value of $\widetilde{Z}_L(1,\ldots,1)$ is equal to the value of the usual continuous time TASEP normalization factor, which matches with the previous value (see (2.43)). Hence we have $\widetilde{Z}_L(1,\ldots,1) = s^C_{1^L}(1,\ldots,1|\mathbf{v})$, so $f=0$. This concludes the proof.

B Expansion of shifted Schur polynomials of type C

We show that the type C shifted Schur polynomial (2.35) can be expanded on the regular type C Schur polynomials as follows

$$s^C_{1^L}(z_1,\dots,z_L|\mathbf{v}) = \sum_{n=1}^{L+1} \frac{a^n-b^n}{a-b}\, s^C_{1^{L+1-n}}(z_1,\dots,z_L), \tag{B.1}$$

where $\mathbf{v}=(-a,-b,0,\dots,0)$.

Proof. Starting from

$$s^C_{1^L}(z_1,\dots,z_L|\mathbf{v}) = \frac{\det\left((z_j|\mathbf{v})^{L+2-i} - (1/z_j|\mathbf{v})^{L+2-i}\right)_{i,j}}{\det\left(z_j^{L+1-i} - (1/z_j)^{L+1-i}\right)_{i,j}}$$

the general idea is to use the multi-linearity of the determinant in the numerator to expand it and recognize non shifted Schur polynomials of type C. First we rewrite

$$\begin{aligned}(z_j|\mathbf{v})^{L+2-i} - (1/z_j|\mathbf{v})^{L+2-i} &= z_j^{L+2-i} - \frac{1}{z_j^{L+2-i}} + (a+b)\left(z_j^{L+1-i} - \frac{1}{z_j^{L+1-i}}\right) \\ &\quad + ab\left(z_j^{L-i} - \frac{1}{z_j^{L-i}}\right).\end{aligned} \tag{B.2}$$

Define the matrix $M(z_1,\dots,z_L) = \left((z_j|\mathbf{v})^{L+2-i} - (1/z_j|\mathbf{v})^{L+2-i}\right)_{i,j}$. Let $R_i(z_1,\dots,z_L)$ be the i-th row of $M(z_1,\dots,z_L)$. The expansion (B.2), we can rewritten as

$$R_i(z_1,\dots,z_L) = P_{i-1}(z_1,\dots,z_L) + (a+b)P_i(z_1,\dots,z_L) + abP_{i+1}(z_1,\dots,z_L), \tag{B.3}$$

with $P_k(z_1,\dots,z_L)$ the row vector $(z_j^{L+1-k} - 1/z_j^{L+1-k})_j$. Remark that $P_{L+1}(z_1,\dots,z_L) = 0$. We develop $\det(M(z_1,\dots,z_L))$ using the expansion (B.3) and the multi-linearity of the determinant. Since we have only $L+1$ different row vectors P_i, antisymmetry of the determinant imposes that we need to choose L distinct row vectors P_i among $P_0,\dots,P_L$ in order to get a non vanishing determinant (hence there is just one P_i missing in each non vanishing determinant). Then, the expansion of $\det(M(z_1,\dots,z_L))$ reads

$$\det(M(z_1,\dots,z_L)) = \sum_{n=1}^{L+1} C_n\, u_n(z_1,\dots,z_L),$$

where

$$u_{L+1-k}(z_1,\dots,z_L) = \det\begin{pmatrix} P_0(z_1,\dots,z_L) \\ \vdots \\ P_{k-1}(z_1,\dots,z_L) \\ P_{k+1}(z_1,\dots,z_L) \\ \vdots \\ P_L(z_1,\dots,z_L)\end{pmatrix}, \qquad \text{with} \qquad k\in\{0,..,L\} \tag{B.4}$$

and C_n are some coefficients to be determined.

Let us first compute C_1 (which corresponds to the term with missing P_L in (B.4)). There is only one possibility to get the row vector P_0: it comes necessarily from the row R_1. Then P_1 appears in the rows R_1 and R_2, but since we have already chosen P_0 for the row R_1, P_1 has to be taken in row R_2. One shows by induction that for all $i \in \{0, \ldots L-1\}$, P_i has to be taken in row R_{i+1}. It is summarized on the following diagram.

$$\begin{array}{ccccccc}
R_1 & \mathbf{P_0} & P_1 & P_2 & & & \\
R_2 & & \mathbf{P_1} & P_2 & P_3 & & \\
\vdots & & & \ddots & \ddots & \ddots & \\
R_{L-1} & & & & \mathbf{P_{L-2}} & P_{L-1} & \cancel{P_L} \\
R_L & & & & & \mathbf{P_{L-1}} & \cancel{P_L}
\end{array} \tag{B.5}$$

Each line in (B.5) corresponds to the expansion (B.3) of a row vector R_i. The coefficients have been omitted, but can be easily recovered from (B.3). The bold P_i's in the diagram are the ones chosen in the expansion to obtain the term $u_1(z_1, \ldots, z_L)$. The shaded P_i's are the ones that do not appear in $u_1(z_1, \ldots, z_L)$. In the case considered here, all the chosen P_i's correspond to the first term in (B.3) and hence come with a factor 1. We deduce that $C_1 = 1$.

The computation of C_2 (with P_{L-1} missing) looks similar. There is again only one choice for the P_i's to get $u_2(z_1, \ldots, z_L)$. It is given by

$$\begin{array}{cccccccc}
R_1 & \mathbf{P_0} & P_1 & P_2 & & & & \\
R_2 & & \mathbf{P_1} & P_2 & P_3 & & & \\
\vdots & & & \ddots & \ddots & \ddots & & \\
R_{L-2} & & & & \mathbf{P_{L-3}} & P_{L-2} & \cancel{P_{L-1}} & \\
R_{L-1} & & & & & \mathbf{P_{L-2}} & \cancel{P_{L-1}} & P_L \\
R_L & & & & & & \cancel{P_{L-1}} & \mathbf{P_L}
\end{array} \tag{B.6}$$

The chosen $P_0, \ldots, P_{L-2}$ correspond to the first term in (B.3) and hence come with a factor 1, but the chosen P_L corresponds to the second term in (B.3) and thus comes with a factor $a+b$. We deduce that $C_2 = a + b$.

To compute C_n (with P_{L+1-n} missing), we draw the diagram

$$\begin{array}{ccccccccccc}
R_1 & \mathbf{P_0} & P_1 & P_2 & & & & & & & \\
R_2 & & \mathbf{P_1} & P_2 & P_3 & & & & & & \\
\vdots & & & \ddots & \ddots & \ddots & & & & & \\
R_{L-n} & & & & \mathbf{P_{L-1-n}} & P_{L-n} & \cancel{P_{L+1-n}} & & & & \\
R_{L+1-n} & & & & & \mathbf{P_{L-n}} & \cancel{P_{L+1-n}} & P_{L+2-n} & & & \\
R_{L+2-n} & & & & & & \cancel{P_{L+1-n}} & P_{L+2-n} & P_{L+3-n} & & \\
R_{L+3-n} & & & & & & & P_{L+2-n} & P_{L+3-n} & P_{L+4-n} & \\
\vdots & & & & & & & & \ddots & \ddots & \ddots
\end{array} \tag{B.7}$$

In opposition to the previous cases, there is some freedom in this diagram. Indeed, although the first rows $R_1, \dots, R_{L+1-n}$ uniquely fix which P_i's are chosen (written in bold), the row R_{L+2-n} corresponds to two choices:

- we choose P_{L+2-n} (which comes with the factor $a+b$). Then, the remaining rows just correspond to the computation of C_{n-1}.
- we choose P_{L+3-n} (which comes with the factor ab). In this case, we have to choose P_{L+2-n} in the row R_{L+3-n} (which comes with the factor 1). Then, the remaining rows correspond to the computation of C_{n-2} (except that the order of P_{L+2-n} and P_{L+3-n} is inverted, hence a factor (-1) by antisymmetry of the determinant).

Altogether, we get a recursion relation for the coefficients C_n:

$$C_n = (a+b)\, C_{n-1} - ab\, C_{n-2}. \tag{B.8}$$

It is straightforward to check that $C_n = \frac{a^n - b^n}{a-b}$ verifies (B.8), $C_1 = 1$ and $C_2 = a+b$. We complete the proof of (B.1) by remarking that $u_n(z_1, \dots, z_L)/\det\left(z_j^{L+1-i} - (1/z_j)^{L+1-i}\right)_{i,j} = s^C_{1^{L+1-n}}(z_1, \dots, z_L)$.

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